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# Large sample properties of Gibbs-type priors

Pierpaolo De Blasi, Antonio Lijoi and Igor Prünster

**Abstract** In this paper we concisely summarize some recent findings that can be found in [1] and concern large sample properties of Gibbs-type priors. We shall specifically focus on consistency according to the frequentist approach which postulates the existence of a “true” distribution  $P_0$  that generates the data. We show that the asymptotic behaviour of the posterior is completely determined by the probability of obtaining a new distinct observation. Exploiting the predictive structure of Gibbs-type priors, we are able to establish that consistency holds essentially always for discrete  $P_0$ , whereas inconsistency may occur for diffuse  $P_0$ . Such findings are further illustrated by means of three specific priors admitting closed form expressions and exhibiting a wide range of asymptotic behaviours.

**Key words:** Asymptotics, Bayesian nonparametrics, Gibbs-type priors

## 1 Gibbs-type priors

In this paper we sketch results that are extensively presented and proved in [1] about the asymptotic posterior behaviour of Gibbs-type priors, a class of discrete nonparametric priors recently introduced in [5]. Gibbs-type priors can be defined through the system of predictive distributions they induce. To this end, let  $(X_n)_{n \geq 1}$  be an (ideally) infinite sequence of observations, with each  $X_i$  taking values in a complete and separable metric space  $\mathbb{X}$ . Moreover,  $\mathbf{P}_{\mathbb{X}}$  is the set of all probability measures on  $\mathbb{X}$  endowed with the topology of weak convergence. In the most commonly employed Bayesian models  $(X_n)_{n \geq 1}$  is assumed to be *exchangeable* which means there exists a

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probability distribution  $Q$  on  $\mathbf{P}_{\mathbb{X}}$  such that  $X_i|\tilde{p} \stackrel{\text{iid}}{\sim} \tilde{p}$ ,  $\tilde{p} \sim Q$ . Hence,  $\tilde{p}$  is a random probability measure on  $\mathbb{X}$  whose probability distribution  $Q$  is also termed *de Finetti measure* and acts as a prior for Bayesian inference. Given a sample  $(X_1, \dots, X_n)$ , the predictive distribution coincides with the posterior expected value of  $\tilde{p}$ , that is

$$\text{pr}(X_{n+1} \in \cdot | X_1, \dots, X_n) = \int_{\mathbf{P}_{\mathbb{X}}} p(\cdot) Q(dp | X_1, \dots, X_n). \quad (1)$$

A prior  $Q$  that selects, almost surely, discrete distributions is said discrete and, in this case, a sample  $(X_1, \dots, X_n)$  will feature ties with positive probability:  $X_1^*, \dots, X_k^*$  denote the  $k \leq n$  distinct observations and  $n_1, \dots, n_k$  their frequencies for which  $\sum_{i=1}^k n_i = n$ . Gibbs-type priors are discrete and characterized by predictive distributions (1) of the form

$$\text{pr}(X_{n+1} \in \cdot | X_1, \dots, X_n) = \frac{V_{n+1,k+1}}{V_{n,k}} P^*(\cdot) + \frac{V_{n+1,k}}{V_{n,k}} \sum_{i=1}^k (n_i - \sigma) \delta_{X_i^*}(\cdot), \quad (2)$$

where  $\sigma \in (-\infty, 1)$ ,  $P^*(dx) := E[\tilde{p}(dx)]$  is a diffuse probability measure representing the prior guess at the shape of  $\tilde{p}$  and  $\{V_{n,k} : k = 1, \dots, n; n \geq 1\}$  is a set of non-negative weights satisfying the recursion

$$V_{n,k} = (n - \sigma k) V_{n+1,k} + V_{n+1,k+1}. \quad (3)$$

Therefore, Gibbs-type priors are characterized by predictive distributions, which are a linear combination of the prior guess and a weighted version of the empirical measure. The most widely known priors within this class are the Dirichlet process [3] and the two-parameter Poisson-Dirichlet process [8].

## 2 Consistency results

We address posterior consistency according to the “what if” approach of [2], which consists in assuming that the data  $(X_n)_{n \geq 1}$  are independent and identically distributed from some “true”  $P_0 \in \mathbf{P}_{\mathbb{X}}$  and in verifying whether the posterior distribution  $Q(\cdot | X_1, \dots, X_n)$  accumulates in any neighborhood of  $P_0$ , under a suitable topology. Since Gibbs-type priors are defined on  $\mathbf{P}_{\mathbb{X}}$  and are discrete, the appropriate notion of convergence is convergence in the weak topology. Therefore, we aim at establishing whether  $Q(A_\varepsilon | X_1, \dots, X_n) \rightarrow 1$ , a.s.- $P_0^\infty$ , as  $n \rightarrow \infty$  and for any  $\varepsilon > 0$ , where  $A_\varepsilon$  denotes a weak neighborhood of some  $P' \in \mathbf{P}_{\mathbb{X}}$  of radius  $\varepsilon$  and  $P_0^\infty$  is the infinite product measure. Clearly, consistency corresponds to  $P' = P_0$ .

We prove a general structural result on Gibbs-type priors showing that the posterior distribution converges to a point mass at the weak limit, in an almost sure sense, of the predictive distribution (2). To this aim, we shall assume that the probability of recording a new distinct observation at step  $n+1$

$$\frac{V_{n+1, \kappa_n+1}}{V_{n, \kappa_n}} \quad \text{converges} \quad \text{a.s.}-P_0^\infty \quad (\text{H})$$

as  $n \rightarrow \infty$ , and the limit is identified by some constant  $\alpha \in [0, 1]$ . We use the notation  $\kappa_n$  in order to make explicit the dependence on  $n$  of the number of distinct observations in a sample of size  $n$ . Different choices of  $P_0$  yield different limiting behaviours for  $\kappa_n$ : if  $P_0$  is discrete with  $N$  point masses, then  $P_0^\infty(\lim_n n^{-1} \kappa_n = 0) = 1$  even when  $N = \infty$ ; if  $P_0$  is diffuse,  $P_0^\infty(\kappa_n = n) = 1$  for any  $n \geq 1$ . The following theorem shows that (H) is actually sufficient to establish weak convergence at a certain  $P'$  that is explicitly identified. The key ingredient for the proof is represented by an upper bound on the posterior variance  $\text{Var}[\tilde{p}(A) | X_1, \dots, X_n]$ , which is of independent interest. See [1].

**Theorem 1.** *Let  $\tilde{p}$  be a Gibbs-type prior with prior guess  $P^* = E[\tilde{p}]$ , whose support coincides with  $\mathbb{X}$ , and assume condition (H) holds true. If  $(X_i)_{i \geq 1}$  is a sequence of independent and identically distributed random elements from  $P_0$  then the posterior converges weakly, a.s.- $P_0^\infty$ , to a point mass at  $\alpha P^*(\cdot) + (1 - \alpha)P_0(\cdot)$ .*

According to Theorem 1, weak consistency is achieved in the trivial case of  $P^* = P_0$ , which will be excluded henceforth, and when  $\alpha = 0$ : therefore, it is sufficient to check whether the probability of obtaining a new observation, given previously recorded data, converges to 0, a.s.- $P_0^\infty$ . One might also wonder whether there are circumstances leading to  $\alpha = 1$ , which corresponds to the posterior concentrating around the prior guess  $P^*$ , a situation we refer to as “total” inconsistency.

Since a few particular cases of Gibbs-type priors with  $\sigma \in (0, 1)$  have already been considered in [7] and [6], attention is focused on the case of  $\sigma \in (-\infty, 0)$  for which nothing is known to date. We recall here that, if  $\sigma < 0$ ,  $Q$  is a mixture of Poisson-Dirichlet processes with parameters  $(\sigma, k|\sigma|)$  and the mixing distribution for  $k$ , say  $\pi$ , is supported by the positive integers. Since in the case of negative  $\sigma$  the two-parameter model coincides with a  $x$ -variate symmetric Dirichlet distribution, one can describe such Gibbs-type priors as

$$\begin{aligned} (\tilde{p}_1, \dots, \tilde{p}_k) &\sim \text{Dirichlet}(|\sigma|, \dots, |\sigma|) \\ k &\sim \pi(\cdot) \end{aligned} \quad (4)$$

We shall restrict attention to Gibbs-type priors whose realizations are discrete distributions whose support has a cardinality that cannot be bounded by any positive constant, almost surely. This is the same as assuming that the support of  $\pi$  in (4) is  $\mathbb{N}$ . Note that for the “parametric” case of  $\sigma < 0$  and  $\pi$  supported by a finite subset of  $\mathbb{N}$  one immediately has consistency for any  $P_0$  in its support by the results of [4]. Theorem 2 gives neat sufficient conditions for consistency in terms of the tail behaviour of the mixing distribution  $\pi$  on the positive integers  $\mathbb{N}$  in (4).

**Theorem 2.** *Let  $\tilde{p}$  be a Gibbs-type prior with parameter  $\sigma < 0$ , mixing measure  $\pi$  and prior guess  $P^*$  whose support coincides with  $\mathbb{X}$ . Then the posterior is consistent*

(i) *at any discrete  $P_0$  if for sufficiently large  $x$*

$$\pi(x+1)/\pi(x) \leq 1; \quad (\text{T1})$$

(ii) at any diffuse  $P_0$  if for sufficiently large  $x$  and for some  $M < \infty$

$$\pi(x+1)/\pi(x) \leq M/x. \quad (\text{T2})$$

Note that condition (T1) is an extremely mild assumption on the regularity of the tail of the mixing  $\pi$ : it requires  $x \mapsto \pi(x)$  to be ultimately decreasing, a condition met by the commonly used probability measures on  $\mathbb{N}$ . On the other hand, condition (T2) requires the tail of  $\pi$  to be sufficiently light. This is indeed a binding condition and it is particularly interesting to note that such a condition is also close to being necessary. In [1], three different Gibbs-type priors with  $\sigma = -1$  are considered, each prior characterized by a specific choice of the mixing distribution  $\pi$ . These examples show that, according as to heaviness of the tails of  $\pi$ , the value of  $\alpha$  in Theorem 1 may actually span the whole interval  $[0, 1]$ , from situations where consistency holds true ( $\alpha = 0$ ) to cases where “total” inconsistency occurs ( $\alpha = 1$ ). In particular, the heavier the tail of  $\pi$  and the larger  $\alpha$ , i.e. the lighter is the weight assigned to the “true”  $P_0$  in the limiting distribution identified in Theorem 1. The first prior is characterized by a heavy-tailed mixing distribution  $\pi$ , which does not admit a finite expected value: condition (T2) is not met and it turns out that  $\alpha = 1$  so that the posterior concentrates around the prior guess  $P^*$  (“total” inconsistency). The second specific prior, where the mixing  $\pi$  has light tails that satisfy (T2) in Theorem 2, results in a consistent asymptotic behaviour. In the third case  $\alpha$  takes values over the whole unit interval  $[0, 1]$  according to a parameter that determines the heaviness of the tail of  $\pi$ .

## References

1. De Blasi, P., Lijoi, A., Prünster, I.: An asymptotics analysis of a class of discrete nonparametric priors. Technical Report (2012)
2. Diaconis, P., Freedman, D.: On the consistency of Bayes estimates. *Ann. Statist.* **14**, 1-26 (1986)
3. Ferguson, T. S.: A Bayesian analysis of some nonparametric problems. *Ann. Statist.* **1**, 209-230 (1973)
4. Freedman, D.: On the asymptotic behavior of Bayes’ estimates in the discrete case. *Ann. Math. Statist.* **34**, 1386-1403 (1963)
5. Gnedin, A., Pitman, J.: Exchangeable Gibbs partitions and Stirling triangles. *Journal of Mathematical Sciences* **138**, 5674-5685 (2006)
6. James, L. F.: Large sample asymptotics for the two parameter Poisson Dirichlet process. In: B. Clarke and S. Ghosal (eds.) *Pushing the Limits of Contemporary Statistics: Contributions in Honor of Jayanta K. Gosh*, pp. 187-199. IMS, Harvard (2008)
7. Jang, G. H., Lee, J., Lee, S.: Posterior consistency of species sampling priors. *Statistica Sinica* **20**, 581-593 (2010)
8. Pitman, J. : Some developments of the Blackwell-MacQueen urn scheme. In: Ferguson, T.S., Shapley, L.S., MacQueen, J.B. (eds.) *Statistics, Probability and Game Theory. Papers in honor of David Blackwell*, pp. 245-267. *Lecture Notes, Monograph Series*, **30**, IMS, Harvard (1996)